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THE LORENTS CONDITION AS A SECONDARY GAUGE CONDITION AND ITS APPLICATION FOR THE FIELD QUANTIZATION

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The quantization of a free electromagnetic field is performed by using a secondary gauge condition. It is proved that in contrast with the standard approach the generating functional of the Green function must contain two δ-functions with the gauge conditions under the sign of the functional integration in configurational space.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Условие Лоренца как вторичное калибровочное условие и его использование при квантовании полей

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Осуществлено квантование свободного электромагнитного поля с учетом вторичного калибровочного условия. Доказано, что, в отличие от стандартного подхода, производящий функционал функций Грина должен содержать под знаком функционального интеграла в конфигурационном пространстве две δ-функции с калибровочными условиями. Это приводит к существенной модификации стандартной диаграммной техники.

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In our previous work /1/ a general theorem was proved that claims that Yang-Mills field A_μ, after imposing on it an arbitrary gauge condition $\Phi(A) = 0$, does satisfy one more complementary condition. This condition in QED has the form **

$$\partial^\mu A_\mu(x) = 0. \qquad (1)$$

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** *Condition (1) holds for a free electromagnetic field that is just quantized in the framework of the perturbation theory. At the presence of the interaction the spinor current enters into the right-hand side of the secondary gauge condition (1) (see 1). In a non-Abelian case (1) is substituted by the condition* $\tilde{\partial}^\mu A_\mu = 0$, *were* $\tilde{\partial}_\mu$ *is the Mandelstam derivative.*

Due to the fact that the relation (1) has been derived allowing for the primary gauge condition $\Phi(A) = 0$ together with the equation of motion it has the sense of the secondary constraint and therefore is named by us as the **secondary gauge condition**. (Let us remind that the secondary constraints by definition [2] are obtained from the primary ones taking account of the equations of motion).

The proof of this general theorem, in [1] is based on the existence of the conditions of field decreasing at infinity that have the from

$$A_\mu(x) \underset{|x|\to\infty}{\sim} \frac{1}{|x|^{1+\epsilon}}; \quad x \equiv \sqrt{x_0^2 - \vec{x}^2}; \quad 0 < \epsilon \leq 1, \tag{2}$$

and that are necessary for combining the requirement of the finite of action with the possibility of using the integration by parts, that is in turn necessary for constructing the perturbation theory.

In [1] we have performed a modification of the known methods of the vector field quantization: the Dirac — Bergmann method and the method of the operator quantization by including the secondary gauge condition (1) into the system of constraints. But the most rigorous quantization method is that of the quantization by a functional integral starting directly from the phase space. In the present paper we shall perform such a quantization taking account of the secondary gauge condition (1). While doing this the difference between our approach and Faddeev — Popov method would become clear. It consists, as it will be shown below, in consistency of our quantization procedure by functional method with the physicas condition on the gauge field asymptotics.

Let us perform the quantization procedure for the free electromagnetic field with the Lagrangian density

$$\mathfrak{L}(x) = -\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x) \quad (F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu).$$

The corresponding density of the canonical Hamiltonian has the form

$$H(x) = \frac{1}{4} F^{ij}(x) F_{ij}(x) - \frac{1}{2} \pi^i(x) \pi_i(x) - A_0(x) \partial^i \pi_i(x), \tag{3}$$

where $\pi^\mu(x) = \partial \mathfrak{L}(x) / \partial \dot{A}_\mu(x)$ is the canonical momentum. In accordance with the general quantization procedure for the constrained systems [3-5] we have the following espression for the matrix element of the S-matrix in the extended phase space Γ^*:

$$<\text{out}|S|\text{in}> \sim \int \prod_{\mu} DA_{\mu} D\pi^{\mu} \delta(\phi_1(A,\pi))\delta(\phi_2(A,\pi)) \times$$

$$\times \delta(\chi_1(A,\pi))\delta(\chi_2(A,\pi)) \cdot \det\|\phi_i, \chi_j\|_{i,j=1,2} \times \tag{4}$$

$$\times \exp[\pi^{\mu} \dot{A}_{\mu} - H_o(A, \pi)],$$

where $\phi_1 = \pi_o$ is the primary constraint, $\phi_2 = \partial^i \pi_i$ is the secondary constraint and χ_1, χ_2 are additional gauge conditions that satisfy the relations $\{\chi_1, \chi_2\} = 0$, $\det\|\phi_i, \chi_j\| \neq 0$.

Let us consider for example the case when the field A_μ obeys the temporal gauge condition $\chi_1 = A_o$ (as the primary gauge condition). Then, due to the general theorem [1] the field A_μ has to obey, on the equations of motion, at the same time the secondary gauge condition (1) that in this particular case, due to the conservation of the condition $\chi_1 = A_o$ in time, takes the form of the Coulomb condition $\chi_2 = \partial^\mu A_\mu = \partial^i A_i$. Thus, the system of the second class constraints has the form **

$\phi_1 = \pi_o \approx 0$ — primary constraint; $\chi_1 = A_o \approx 0$ — primary gauge condition; (5)

$\phi_2 = \partial^i \pi_i \approx 0$ — secondary constraint; $\chi_2 = \partial^i A_i \approx 0$ — secondary gauge condition, (6)

that formally does not differ from an analogous system of constraints used in the framework of the standard approach [4,5] for the case of the so-called "radiational gauge" ($A_o = 0$, $\mathrm{div}\vec{A} = 0$). But there exists an essential difference between our and the standard approaches. Firstly, in the standard spproach one has to use the residual gauge arbitrariness to complete the primary condition $A_o = 0$ by the condition $\mathrm{div}\,\vec{A} = 0$. As it has been shown by us in [1], the residual gauge arbitrariness in the gauge $A_o = 0$ is not compatible with the physical boun-

* *As usually the sing " ~ " means the definition up to the normalization factor that in the general case may appear to be an infinite constant.*

** *The sing "≈" means the equivalence to zero in a weak sense, i.e. after opening the Poisson brackets [2].*

dary conditions (2). There it has been also shown that the condition $\partial^i A_i^T = 0$ is the sequence of the imposing of uniqually achieved gauge condition[1] $A_0 = 0$, on the field A_μ and the Maxwell equations only ($A_\mu^T = A_\mu + \partial_\mu \Lambda^T(A, x)$), where $\Lambda^T(A, x) = \int_{-\infty}^{\rho} d\alpha \cdot A_0(x_0 + \alpha, \vec{x})$ is the projector on the gauge $A_0^T = 0$. The residual gauge arbitrariness is completely absent in this case.

The next important fact follows from this circumstance: the condition $\chi_2 = \partial^i A_i \approx 0$ is the only possible condition that completes the system $\phi_1 = \pi_0 \approx 0$, $\phi_2 = \partial^i \pi_i \approx 0$, $\chi_1 = A_0 \approx 0$ up to the system of the second class constraints.

Taking into account (6) and combining the constant $\det \| \phi_i(t, \vec{x}), \phi_j(t, \vec{y}) \|_{i,j=1,2} = \det \| \delta_{ij} \nabla^2 \delta(\vec{x} - \vec{y}) \|$ with the normalization factor, we shall obtain instead of (4) the relation

$$<\text{out} | S | \text{in}> \sim \int DA^\mu D\pi_\mu \cdot \delta(\pi_0) \cdot \delta(\partial^i \pi_i) \delta(A_0) \times \\ \times \delta(\partial^i A_i) \cdot \exp i [\pi^i \dot{A}_i - H_0(A, \pi)], \tag{7}$$

Formula (7) itself does not lead to physical consequences. In order to develop the diagram technique, one has to pass to the configurational space performing the integration over the canonical momenta π. Just at this principal step there appears the main difference between our approach and the standard method[3].

Let us remind that in the standard approach the integrals over π_0 and A_0 are easily taken with the help of $\delta(\pi_0)$ and $\delta(A_0)$ functions and then the measure $\prod_i DA_i$ is reconstructed up to the complete integration measure $\prod_\mu DA_\mu$ with the help of the integral representation of $\delta(\partial^i \pi_i)$: $\delta(\partial^i \pi_i) = \int DV \exp[i \int d^4x V(x) \partial^i \pi_i]$. Thus as a result of the Gauss integration over $\vec{\pi}$, one obtains in the standard approach the following formula:

$$<\text{out} | S | \text{in}> \sim \int \prod_\mu DA_\mu \cdot \delta(\partial^i A_i) \exp[i \int d^4x (-\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x))], \tag{8}$$

where A_0 is chosen to be $A_0 \equiv V$.

We consider this prescription to be logically inconsistent because it is based on the substitution of the truly dynamical variable A_0 (that is canonically conjugated to the momentum π_0 and satisfies the physical boundary condition (1)) by the completely arbitrary function V. It leads to the necessity of the correct account of the Gauss law while integration over the canonical momenta. For this purpose we shall use the well-known integral representation for the functional δ-function

$$\delta(f[A]) \equiv \prod_x \delta(f[A(x)]) = \lim_{a\to o} \prod_x \frac{1}{\sqrt{-2i\pi a}} \exp(-\frac{i}{2a}\{f[A(x)]\}^2) \sim \tag{9}$$

$$\sim \lim_{a\to o} \exp(-\frac{i}{2a} \int d^4x \{ f[A(x)] \}^2).$$

After performing in (7) the integration over π_0 and using the representation (9) for $\delta(\partial^i \pi_i)$ let us perform the Gauss integration over $\vec{\pi}$. We obtain as a result

$$< out | S | in > \sim \int \prod_\mu DA_\mu \, \delta(A_o) \, \delta(\partial^i A_i) \times \tag{10}$$

$$\times \exp[-i \int d^4x \frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x)] \exp[-\frac{i}{2} \int d^4x d^4y \dot{A}^i(x) K_{ij}(x-y) \dot{A}^j(y)];$$

where the Fourier transform $\tilde{K}_{ij}(\vec{p})$ of the kernel $K_{ij}(x-y)$ of the quadratic form is

$$\tilde{K}_{ij}(\vec{p}) = \delta_{ij} - \frac{p_i p_j}{\vec{p}^2}. \tag{11}$$

Due to the condition $\partial^i A_i = 0$ and the boundary conditions (2), the second term in the right-hand side of (11) gives the zero contribution to (10). Taking account of this circumstance and the fact that due to the condition $A_o = 0$ (that must hold at any world point) the next relation $\partial^i A_o = 0$ takes place, we obtain the next final expression

$$< out | S | in > \sim \int \prod_\mu DA_\mu \, \delta(A_o) \, \delta(\partial^i A_i) \exp[i \int d^4x (-\frac{1}{4} F^{\mu\nu}(x) F_{\mu\nu}(x))]. \tag{12}$$

In (12) in contrast with the standard relation (8) two δ-functions $\delta(A_0)$ and $\delta(\partial^i A_i)$ are put together under the integration sign, which leads to the principle difference of the diagram technical appearing in our approach from the standard one. Really, due to (12) the generation functional of the photon Green functions (that are free in the limit of the sources $J^\mu \to 0$) would have the form

$$G[J] \sim \int \prod_\mu DA_\mu \, \delta(n^\mu A_\mu) \delta(\partial^\mu A_\mu) \times$$

$$\times \exp\{i\int d^4x[-\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) + J^{\mu}(x)A_{\mu}(x)], \qquad (13)$$

where we have introduced the vector $n = (1,0,0,0)$ to give a covariant form for the formula. With the help of representation (9) for the functional δ-fanctions $\delta(n^{\mu}A_{\mu})$ and $\delta(\partial^{\mu}A_{\mu})$ we get in a usual way* /6/

$$G[J] = \exp[-\frac{i}{2}\int d^4x d^4y\, J^{\mu}(x)\Delta^{tr}_{\mu\nu}(x-y)J^{\nu}(y)], \qquad (14)$$

where the Fourier transform of the propagator $\Delta^{tr}_{\mu\nu}(x)$ has the form

$$\tilde{\Delta}^{tr}_{\mu\nu}(p) = -\frac{1}{p^2}\{g_{\mu\nu} + \frac{n^2 p_{\mu}p_{\nu} - (np)(n_{\mu}p_{\nu} + n_{\nu}p_{\mu}) + p^2 n_{\mu}n_{\nu}}{(np)^2 - p^2 n^2}\},$$

that coincides with the propagator obtained by us earlier in /1/ by the method of the operator quantization. But the method of the quantization, given in this article, is the most rigorous because it starts directly from the physical phase space. So, here a rigorous justification of the new approach /1/ to the quantization of gauge fields is given.

Our next publications would be devoted to the application of the propagators obtained in this approach and to the generalization of the method to the non-Abelian case.

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* *For this purpose the ordinary normalization condition* $G(0) = 1$ *is used.*

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